

Thermodynamics of nonlinear bolometers near equilibrium

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Thermodynamics of nonlinear bolometers near equilibrium [★]

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Abstract

We present the first thermodynamically correct calculation of the noise in a simple nonlinear resistive bolometer or calorimeter operated out of equilibrium. The solution is rigorous only for first- and second-order deviations from equilibrium, and for the linear and quadratic terms of dissipative elements. In contrast, existing models of noise in resistive bolometers are based on the application of equilibrium theories to a system that is often nonlinear and out of equilibrium. We derive solutions applicable both in and out of steady state. The noise has power spectral density different from the equilibrium theory, and it has higher-order correlations and non-Gaussian characteristics. The results do not apply to non-Markovian hidden variables in the bolometer.

Key words: bolometers, microcalorimeters, nonlinear thermodynamics, nonequilibrium

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1. Introduction

The equilibrium thermodynamics of resistive bolometers and calorimeters is well established [1,2]. However, these calculations are not rigorous in non-equilibrium systems (in which the bolometer temperature is different from the bath temperature). An analysis of the noise of a simple nonlinear, nonequilibrium bolometer is a useful step towards understanding and optimizing the noise in a real-world bolometer or calorimeter. For instance, in the case of the superconducting transition-edge sensor (TES), noise in excess of equilibrium sources is usually observed, and is of-

ten labeled as “excess noise.” Most attempts to explain this noise focus on the complicated, non-Markovian details of the superconducting system. However, it is beneficial to develop a better understanding of the baseline thermodynamic noise of a simple system to better understand what is meant by “excess.”

2. Quadratic Markovian bolometers

We assume a simple Markovian bolometer (i.e., with no hidden variables such as internal temperature gradients and fluctuating current paths). We consider an external voltage bias, which we model as an internal capacitor in series with the bolometer (Fig. 1) with capacitance $C \rightarrow \infty$. The system is thus closed and

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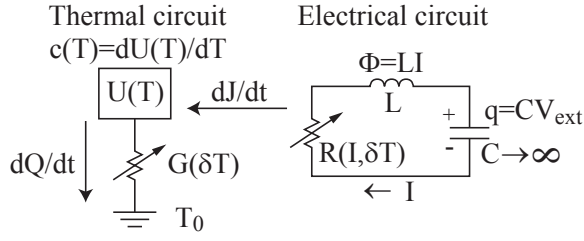


Fig. 1. A model of a voltage-biased resistive bolometer. The thermal circuit consists of a thermal mass at temperature T with energy $U(T)$ and heat capacity $c(T)$, a temperature-dependent thermal conductance $G(\delta T)$, and a heat reservoir at fixed temperature T_0 , where $\delta T \equiv T - T_0$. A heat flow dQ/dt crosses the thermal conductance to the heat reservoir. The electrical circuit consists of a resistor with both current and temperature dependence $R(I, \delta T)$, an inductance L with current I and flux $\Phi = LI$, and a capacitor $C \rightarrow \infty$ with “external” voltage V_{ext} and charge $q = CV_{ext}$. Joule heating in the resistor, dJ/dt , is dissipated in the thermal mass.

time-reversal invariant. Elsewhere, we will consider a current biased bolometer modeled with a parallel infinite inductance.

The electrical resistance R and thermal conductance G are nonlinear functions of I and δT , both of which are small near equilibrium. If R and G are analytic at equilibrium we can express them as a Taylor series expansion to the first order:

$$R(I, \delta T) \equiv \frac{V}{I} \approx \rho_1 + \rho_2 I + \zeta \delta T, \quad (1)$$

$$G(\delta T) \equiv \frac{dQ/dt}{\delta T} \approx \gamma_1 + \gamma_2 \delta T. \quad (2)$$

The macroscopic relaxation equations describing the evolution of the system in Fig. 1 are

$$L \frac{dI}{dt} = -\rho_1 I - \rho_2 I^2 - \zeta I \delta T - \frac{q}{C} \quad (3)$$

$$\frac{dq}{dt} = I \quad (4)$$

$$\frac{dU}{dt} = -\gamma_1 \delta T - \gamma_2 \delta T^2 - I \frac{d\Phi}{dt} - \frac{q}{C} \frac{dq}{dt}. \quad (5)$$

We choose to consider the system as a microcanonical ensemble rather than a canonical ensemble to facilitate the analysis of a nonequilibrium model with different temperatures T and T_0 . As state variables, we choose the flux in the inductor $x_\Phi \equiv \Phi = LI$, the charge on the capacitor $x_q \equiv q$, and the thermal energy of the bolometer, $x_U \equiv U$. The first step in the analysis is to determine the entropy change. The heat flow dQ transfers energy from the bolometer at

temperature T to the heat reservoir at T_0 with entropy change $dS = (1/T_0 - 1/T)dQ$. The energy that flows from the capacitor and inductor, dJ , dissipates power into the bolometer at temperature T , with $dS = dJ/T = -(1/T)(\Phi/L)d\Phi - (1/T)(q/C)dq$. The change in bolometer energy is $dU = dJ - dQ$, so the total change in entropy is

$$dS = \left(\frac{1}{T} - \frac{1}{T_0} \right) dU - \frac{1}{T_0} \frac{\Phi}{L} d\Phi - \frac{1}{T_0} \frac{q}{C} dq. \quad (6)$$

In a microcanonical ensemble, the thermodynamic force, X_α , conjugate to the state variables x_α is [3] $X_\alpha = -\partial S / \partial x_\alpha$, where $\alpha = \Phi, q, U$. The thermodynamic forces conjugate to our state variables are $X_\Phi = \Phi / T_0 L = I / T_0$, $X_q = q / (T_0 C) = V_{ext} / T_0$, and $X_U = (1/T_0 - 1/T) = \delta T / (T_0 T)$.

If all functions in the relaxation equations are analytic at equilibrium, the time derivatives of the state variables can be expressed as a series expansion of the thermodynamic forces. Near equilibrium, the thermodynamic forces are small, and we carry the expansion out to the second order:

$$\frac{dx_\alpha}{dt} \approx L_{\alpha\beta} X_\beta + \frac{1}{2} L_{\alpha\beta\gamma} X_\beta X_\gamma, \quad (7)$$

where summation is carried out over repeated indices. The constant term is zero since the expansion is around equilibrium, and the analysis is rigorous up to quadratic deviations from equilibrium.

To express the relaxation in this form, we use the expressions for the thermodynamic forces to eliminate the state variables and their derivatives from the right hand sides of (3) - (5). We drop all terms above the quadratic in any combination of the thermodynamic forces, arriving at

$$\frac{d\Phi}{dt} = -\rho_1 T_0 X_\Phi - \rho_2 T_0^2 X_\Phi^2 - T_0 X_q - \zeta T_0^3 X_\Phi X_U \quad (8)$$

$$\frac{dq}{dt} = T_0 X_q \quad (9)$$

$$\frac{dU}{dt} = T_0^2 X_\Phi^2 \rho_1 - \gamma_1 T_0^2 X_U - (\gamma_1 T_0^3 + \gamma_2 T_0^4) X_U^2. \quad (10)$$

In these equations, $L_{q,\Phi} = -L_{\Phi,q} = T_0$, in agreement with the Onsager-Casimir reciprocal relations [4] $L_{\alpha\beta} = \epsilon_\alpha \epsilon_\beta L_{\beta\alpha}$, where the time reversal parity of the variables is $\epsilon_\Phi = -1, \epsilon_q = 1, \epsilon_U = 1$.

The fluctuations of the state variables x_α can be characterized by their correlations. In the quadratic approximation, only one-fold, two-fold, and three-fold

correlators are nonzero. The correlators are related to the average fluctuations in the state variables over time $\tau \rightarrow 0$:

$$K_\alpha \equiv \lim_{\tau \rightarrow 0} (\tau^{-1} \langle \Delta x_\alpha \rangle) \quad (11)$$

$$K_{\alpha\beta} \equiv \lim_{\tau \rightarrow 0} (\tau^{-1} \langle \Delta x_\alpha \Delta x_\beta \rangle) \quad (12)$$

$$K_{\alpha\beta\gamma} \equiv \lim_{\tau \rightarrow 0} (\tau^{-1} \langle \Delta x_\alpha \Delta x_\beta \Delta x_\gamma \rangle) . \quad (13)$$

We use Stratonovich's nonequilibrium Markovian fluctuation-dissipation relations [3] to derive the correlators from the $L_{\alpha,\beta}$ and $L_{\alpha,\beta\gamma}$ parameters in (8) - (10), dropping terms of higher order than quadratic. These relations are derived assuming only Markovian processes, time-reversal symmetry, and consistency with equilibrium thermodynamics. The result is seven unique nonzero correlators:

$$K_q = \frac{\Phi}{L} = I \quad (14)$$

$$K_\Phi = -\frac{q}{C} - \rho_1 I - \rho_2 I^2 - \zeta I \delta T + k_B T_0 \frac{\rho_2}{L} \quad (15)$$

$$K_U = \rho_1 I^2 - \gamma_1 \delta T - \gamma_2 \delta T^2 + k_B T_0 \left(-\frac{\rho_1}{L} + \frac{\gamma_1 + \gamma_2 T_0}{c(T)} \right) \quad (16)$$

$$K_{\Phi\Phi} = 2k_B T \rho_1 + 6k_B T_0 \rho_2 I + 2k_B T_0 \zeta \delta T \quad (17)$$

$$K_{UU} = 2k_B T_0 (\gamma_1 T + \gamma_2 T_0 \delta T) \quad (18)$$

$$K_{U\Phi} = K_{\Phi U} = -2k_B T_0 \rho_1 I \quad (19)$$

$$K_{\Phi\Phi\Phi} = -12(k_B T_0)^2 \rho_2 . \quad (20)$$

3. Conclusions

We will present a detailed analysis of (14)-(20) elsewhere; here we make the brief comments that space allows. First, the noise is weakly non-Gaussian because of the threefold correlator (20).

We can write nonlinear Ito-Langevin equations

$$\frac{dx_\alpha(t)}{dt} = K_\alpha(x) + u_{\alpha\delta}(x) \xi_\delta(t) , \quad (21)$$

where $\xi_\delta(t)$ are delta-function-correlated stochastic forces with zero mean. From (21), (16) and (15), for $C \rightarrow \infty$, a nonequilibrium steady-state exists with a stable $\langle I \rangle$ and $\langle \delta T \rangle$. The small terms proportional to $k_B T_0$ cancel the rectification of equilibrium thermodynamic noise.

From (21) and (17), the power spectral density (PSD) of the Langevin voltage fluctuations $V(t) \equiv u_{\Phi\delta}(x(t)) \xi_\delta(t)$ is

$$S_V = 4k_B T \rho_1 + 12k_B T_0 \rho_2 I + 4k_B T_0 \zeta \delta T , \quad (22)$$

which, for $\delta T \rightarrow 0$, reduces to Stratonovich's expression [3] for the voltage noise PSD in a quadratic resistor. If $\alpha_I \equiv \partial(\log(R))/\partial(\log(T))$ and $\beta_I \equiv \partial(\log(R))/\partial(\log(I))$, then, dropping second-order terms,

$$S_V = 4k_B T R (1 + 2\beta_I) , \quad (23)$$

which is independent of α_I to first order. Near equilibrium, (23) confirms the ansatz of [5] for the voltage noise PSD of a nonlinear bolometer.

From (18) the power noise is

$$S_P = 4k_B T_0 (\gamma_1 T + \gamma_2 T_0 \delta T) , \quad (24)$$

which is consistent with the quadratic expansion around equilibrium of the expressions in the literature for phonon noise PSD in both the diffuse [1] and specular scattering cases [6].

It will be interesting to extend the theory to expansions of higher order than the quadratic, where S_V may depend on α_I . Dissipationally undeterminable parameters appear in higher order expansions that prevent a full solution solely from the dissipative properties of the system. However, it may be possible to achieve insight into the excess voltage noise that is observed as a function of α_I in a TES bolometer [7] by fitting the experimental data to these expansions and treating the dissipationally undeterminable parameters as fitting parameters.

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